

Automorphism groups of toroidal horospherical varieties

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We work over the field \mathbb{C} of complex numbers and mainly consider smooth complete varieties.

Motivation

Theorem (Matsushima 1957)

If a Fano manifold X admits a Kähler–Einstein metric, then $\text{Aut}^0(X)$ is reductive.

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Determine the reductivity of $\text{Aut}^0(X)$ for a given variety X .

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- For a given toric variety X , the reductivity of $\mathrm{Aut}^0(X)$ can be easily computed in terms of **Demazure roots**.

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- Rational homogeneous spaces, wonderful varieties, ...
- K3 surfaces, varieties of general type, ...
- For a given toric variety X , the reductivity of $\mathrm{Aut}^0(X)$ can be easily computed in terms of **Demazure roots**.

Goal

Generalize the notion of Demazure roots for toroidal horospherical varieties.

Outline

- 1 Automorphism groups
- 2 Demazure roots for toric varieties
- 3 Toroidal horospherical varieties
- 4 Main results
- 5 Applications to projective space bundles over a raional homogeneous space

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Automorphism groups of complete varieties

Let X be a complete variety. Then $\text{Aut}(X)$ is a smooth group scheme, locally of finite type, so we have

$$1 \rightarrow \text{Aut}^0(X) \rightarrow \text{Aut}(X) \rightarrow \pi_0(\text{Aut}(X)) \rightarrow 1$$

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By [Barsotti 1955, Chevalley 1960], we have

$$1 \rightarrow \mathrm{Aut}_{\mathrm{lin}}^0(X) \rightarrow \mathrm{Aut}^0(X) \xrightarrow{\mathrm{Alb}} \mathrm{Aut}_{\mathrm{ab}}^0(X) \rightarrow 1$$

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- If X is rationally connected, then $\text{Aut}^0(X)$ is linear.

By Levi decomposition,

$$\text{Aut}^0(X) = R_u(\text{Aut}^0(X)) \rtimes L$$

where $R_u(\text{Aut}^0(X))$ is the unipotent radical of $\text{Aut}^0(X)$ and L is a reductive group.

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- 2 Demazure roots for toric varieties**
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Demazure roots of toric varieties

Let F be a smooth complete S -toric variety where $S \cong (\mathbb{C}^*)^{\dim F}$ is the big torus.

Definition

Let F be a toric variety and Σ be the corresponding fan in $N_{\mathbb{R}}$. Let

$$\mathcal{R}(F) = \{m \in M \mid \exists \rho \in \Sigma(1) : \langle m, v_{\rho} \rangle = -1 \ \& \ \langle m, v_{\rho'} \rangle \geq 0 \ \forall \rho' \in \Sigma(1) \setminus \{\rho\}\}$$

where v_{ρ} and $v_{\rho'}$ denote the primitive generator of the ray ρ and ρ' , resp. An element of $\mathcal{R}(F)$ is called a **Demazure root** for Σ .

- $\mathcal{R}(F)$ depends only on $\Sigma(1)$.
- $|\mathcal{R}(F)|$ = the number of relative interior lattice points of the moment polytope

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Definition

- $\mathcal{S}(F) = \{m \in \mathcal{R}(F) : -m \in \mathcal{R}(F)\}$
- $\mathcal{U}(F) = \mathcal{R}(F) \setminus \mathcal{S}(F)$

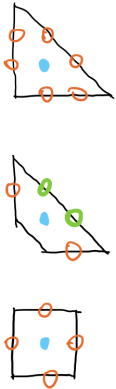
Elements of $\mathcal{S}(F)$ and $\mathcal{U}(F)$ are called **semisimple** and **unipotent**, respectively.

Example

$$\text{fan} \subset N_{\mathbb{R}} = \mathbb{R}^2$$

$$(\text{Moment polytope}) \subset M_{\mathbb{R}} = \mathbb{R}^2$$

\mathbb{P}^2
 \uparrow
 \mathbb{F}_1
 \mathbb{F}_0



Normalized \mathbb{G}_a -actions

Definition

Let L be a linear algebraic group, $\omega \in \text{Hom}(L, \mathbb{C}^*)$ be a character of L , and Y be an L -variety.

- A \mathbb{G}_a -action $A : \mathbb{G}_a \times Y \rightarrow Y$ is L -normalized with weight ω if

$$l.A(s, l^{-1}.y) = A(\omega(l)s, y)$$

for every $l \in L, s \in \mathbb{G}_a, y \in Y$.

- A subgroup H of L is a \mathbb{G}_a -subgroup of L if $H \cong \mathbb{G}_a$.
- An L -root subgroup is the \mathbb{G}_a -subgroup of L associated to the L -normalized \mathbb{G}_a -action A .

Demazure's theorem

Theorem (Demazure)

Let F be a complete S -toric variety. Then

- *$Aut^0(F)$ is generated by S , and S -normalized \mathbb{G}_a -subgroups.*

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- $\text{Aut}^0(F)$ is generated by S , and S -normalized \mathbb{G}_a -subgroups.
- Moreover, there is a bijection between the following sets
 - 1 $\mathcal{R}(F)$
 - 2 the set of all S -normalized \mathbb{G}_a -subgroups of $\text{Aut}^0(F)$.

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 - 1 $\mathcal{R}(F)$
 - 2 the set of all S -normalized \mathbb{G}_a -subgroups of $\text{Aut}^0(F)$.
- Let U_m be the S -root subgroup of $\text{Aut}^0(X)$ corresponding to $m \in \mathcal{R}(F)$.

Demazure roots for toric varieties

Theorem (Demazure)

Let F be a complete S -toric variety.

- 1 The S -root subgroups U_m for $m \in \mathcal{U}(F)$ generate the unipotent radical $R^u(\mathrm{Aut}^0(F))$, and S and U_m for $m \in \mathcal{S}(F)$ generate a Levi subgroup of $\mathrm{Aut}^0(F)$.

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- 2 We have

$$\dim(\text{Aut}^0(F)) = \dim(S) + |\mathcal{R}(F)| = \dim(S) + |\mathcal{S}(F)| + |\mathcal{U}(F)|.$$

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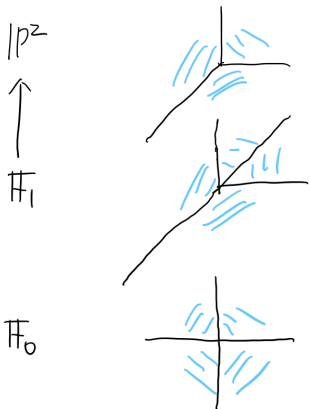
Theorem (Demazure)

Let F be a complete S -toric variety with fan Σ . As S -modules,

$$\text{Lie}(\text{Aut}^0(F)) = \text{Lie}(S) \oplus \bigoplus_{m \in \mathcal{R}(F)} \text{Lie}(U_m).$$

Demazure roots of toric varieties: examples

$$\text{fan} \subset N_{\mathbb{R}} = \mathbb{R}^2 \quad \left(\begin{array}{l} \text{Moment} \\ \text{polytope} \end{array} \right) \subset M_{\mathbb{R}} = \mathbb{R}^2$$

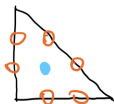


\mathbb{P}^2

↑

\mathbb{F}_1

\mathbb{F}_0



$$\dim \text{Aut}^0(\mathbb{P}^2) = 2 + 6 = 8$$



$$\dim \text{Aut}^0(\mathbb{F}_1) = 2 + 4 = 6$$



$$\dim \text{Aut}^0(\mathbb{F}_0) = 2 + 4 = 6$$

Recall $\text{Aut}(\mathbb{F}_n) \simeq \mathbb{A}^{n+1} \rtimes (\text{GL}_2/\mathbb{G}_m)$ for $n > 0$

$$\text{Aut}(\mathbb{F}_0) \simeq (\text{PGL}_2 \times \text{PGL}_2) \rtimes \{\text{id}, \sigma\}$$

Today's goal

Problem

- *Study the structure of $\mathrm{Aut}^0(X)$ for a toroidal horospherical variety X .*
- *In particular, find the reductivity criterion for $\mathrm{Aut}^0(X)$.*

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Horospherical varieties

Definition

Let G be a connected reductive group.

- 1 A closed subgroup H of G is horospherical if H contains $R^u(B)$ for a Borel subgroup B of G .
- 2 A normal G -variety X is horospherical if X has a dense open subset that is isomorphic to G/H for some horospherical subgroup H of G .

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- 1 (Rational homogeneous spaces) Let P be a parabolic subgroup of G , i.e., it is closed and contains a Borel subgroup. Thus P is horospherical. Then G/P is a horospherical variety.

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Example

- 1 (Rational homogeneous spaces) Let P be a parabolic subgroup of G , i.e., it is closed and contains a Borel subgroup. Thus P is horospherical. Then G/P is a horospherical variety.
- 2 (Toric varieties) Let $G = (\mathbb{C}^*)^n$ be a torus. Take H to be a trivial subgroup of G . Then H is horospherical. Thus any toric variety is horospherical.

Principal torus bundle structure on G/P

Theorem (Pasquier)

Let H be a horospherical subgroup of G and $P = N_G(H)$. Then P is a parabolic subgroup of G and P/H is a torus. Moreover, G/H has a principal torus bundle structure on the rational homogeneous space G/P

$$G/H \rightarrow G/P$$

where the torus fiber is isomorphic to $P/H =: S$.

Toroidal horospherical varieties

Definition

A horospherical variety X is toroidal if every B -stable divisor is G -stable.

- A toroidal horospherical variety X admits a morphism $\pi : X \rightarrow G/P$ extending the natural morphism $G/H \rightarrow G/P$.
- π is a toric bundle over G/P whose S -toric fiber is denoted by F .

Proposition

For every G -horospherical variety X , there exists a G -equivariant birational morphism $\tilde{X} \rightarrow X$ from a toroidal horospherical variety \tilde{X} .

Example

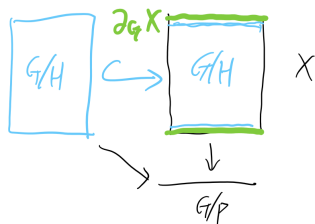
Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. We consider two toroidal horospherical structures.

- 1 X is a $(\mathbb{C}^*)^2$ -horospherical variety, that is, a toric variety.
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- 2 X is a $(SL_2 \times \mathbb{G}_m)$ -horospherical variety.
 - Take $G = SL_2 \times \mathbb{G}_m$.
 - Then $B = B^- = B_{SL_2} \times \mathbb{G}_m$ and $R^u(B) = U_{SL_2} \times \{1\}$.
 - Take $H = B_{SL_2} \times \{1\}$.
 - $P = B_{SL_2} \times \mathbb{G}_m$.
 - So $G/P \cong \mathbb{P}^1$ and $G/H \cong \mathbb{P}^1 \times \mathbb{G}_m$.



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- How to extend the Demazure roots for F to that for X ? (surjectivity)

Demazure roots for horospherical varieties

Definition (-Proposition)

$$\begin{aligned}\mathcal{R}_G^+(X) &= \{m \in \mathcal{R}_S(F) : m \text{ is } B^+ \text{-dominant}\} \\ & (= \{m \in \mathcal{R}_S(F) : \langle m, \epsilon^+(\mathcal{D}^+) \rangle \geq 0\})\end{aligned}$$

Elements of $\mathcal{R}_G^+(X)$ are called Demazure roots (B^+ -roots) of X .

- B^+ is the opposite Borel subgroup of G .
- \mathcal{D}^+ = the set of B^+ -invariant prime divisors on G/H .

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- $\epsilon^+ : \mathcal{D}^+ \rightarrow N_{G/H}$ is the color map defined by $\langle m, \epsilon^+(\mathcal{D}^+) \rangle = \nu_{\mathcal{D}}(f_m)$ for $f_m \in \mathbb{C}(G/H)_m^{(B)} := \{f \in \mathbb{C}(G/H) : b.f = \xi(b)f \ \forall b \in B\}$.

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Theorem (Barban–H–Kwon)

For each Demazure root $m \in \mathcal{R}_S(F)$, the following are equivalent

- 1 $m \in \mathcal{R}_G^+(X)$
- 2 the S -normalized \mathbb{G}_a -action on F associated to m extends to a B^+ -normalized \mathbb{G}_a -action on X preserving every fiber of π .

Reductivity

Definition

- $\mathcal{S}_G^+(X) = \{m \in \mathcal{R}_G^+(X) : -m \in \mathcal{R}_G^+(X)\}$
- $\mathcal{U}_G^+(X) = \mathcal{R}_G^+(X) \setminus \mathcal{S}_G^+(X)$

Elements of $\mathcal{S}_G^+(X)$ and $\mathcal{U}_G^+(X)$ are called semisimple and unipotent, respectively.

Theorem (Barban–H–Kwon)

$\text{Aut}^0(X)$ is reductive if and only if $\mathcal{U}_G^+(X) = \emptyset$.

Decomposition

For $m \in \mathcal{R}_G^+(X)$,

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Theorem (Barban–H–Kwon)

- *As G -modules*

$$\mathrm{Lie}(\mathrm{Aut}^0(X)) = \mathrm{Lie}(\mathrm{Aut}^0(G/P)) \oplus \mathrm{Lie}(\mathrm{Aut}_G(X)) \oplus \bigoplus_{m \in \mathcal{R}_G^+(X)} V(m).$$

- *In particular*

$$\dim \mathrm{Aut}^0(X) = \dim \mathrm{Aut}^0(G/P) + \dim S + |\mathcal{S}_G^+(X)| + \sum_{m \in \mathcal{U}_G^+(X)} \dim V(m).$$

- $\mathrm{Aut}^0(X)$ is generated by $\mathrm{Aut}^0(X, \partial_G X)$ and U_m^+ , for $m \in \mathcal{R}_G^+(X)$.

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- *If the natural morphism $G \rightarrow \mathrm{Aut}^0(G/P)$ is surjective, then L is a Levi subgroup of $\mathrm{Aut}^0(X)$.*
- *If not, X is a toroidal G' -horospherical variety such that $G' \rightarrow \mathrm{Aut}^0(G'/P')$ is surjective where $G' = \mathrm{Aut}^0(X, \partial_G X)$ is the subgroup of $\mathrm{Aut}(X)$ consisting of automorphisms stabilizing every component of $X - (G/H)$.*

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Theorem (Barban–H–Kwon)

- *If the natural morphism $G \rightarrow \mathrm{Aut}^0(G/P)$ is surjective, then L is a Levi subgroup of $\mathrm{Aut}^0(X)$.*
- *If not, X is a toroidal G' -horospherical variety such that $G' \rightarrow \mathrm{Aut}^0(G'/P')$ is surjective where $G' = \mathrm{Aut}^0(X, \partial_G X)$ is the subgroup of $\mathrm{Aut}(X)$ consisting of automorphisms stabilizing every component of $X - (G/H)$.*
- $G \rightarrow \mathrm{Aut}^0(G/P)$ is surjective in most cases.

Historical remarks

- The structure of $\text{Aut}^0(X)$ for a smooth complete spherical variety X is studied in [Bien–Brion96], [Brion07] and [Pezzini09], which was used in our proof.
- $\text{Aut}^0(X)$ for an (quasi-)affine spherical variety X is systematically studied by Avdeev and Zhgoon. In fact, they informed us that they obtained similar results as ours using their preprint arXiv:2312.03377 along with the complete description of the Lie algebra structure, which will be reflected in their manuscript later.

Example

Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. X is a $(SL_2 \times \mathbb{G}_m)$ -horospherical variety.

- Take $G = SL_2 \times \mathbb{G}_m$.
- Then $B = B^- = B_{SL_2} \times \mathbb{G}_m$ and $R^u(B) = U_{SL_2} \times \{1\}$.
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Outline

- 1 Automorphism groups
- 2 Demazure roots for toric varieties
- 3 Toroidal horospherical varieties
- 4 Main results
- 5 Applications to projective space bundles over a raional homogeneous space**

\mathbb{P}^n -bundles over G/P

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Reductivity using Demazure roots

Let P be the parabolic subgroup of G containing B^- such that $Y \simeq G/P$. For each i , let χ_i be the character of P such that $L_i = G \times^P \mathbb{C}_{\chi_i}$.

Theorem (Barban–H–Kwon)

Let $X = \mathbb{P}_Y(L_1 \oplus \cdots \oplus L_k)$. Then

- $\mathcal{R}_G^+(X)$ consists of $\chi_i - \chi_j$ ($i \neq j$) such that $L_i - L_j$ is nef on Y .
- $\mathcal{S}_G^+(X)$ consists of $\chi_i - \chi_j$ ($i \neq j$) such that $L_i \simeq L_j$ as line bundles on Y .

Theorem (Barban–H–Kwon)

Let $X = \mathbb{P}_Y(L_1 \oplus \cdots \oplus L_k)$. Then $\text{Aut}^0(X)$ is reductive if and only if for any i, j with $i \neq j$ such that $L_i \not\simeq L_j$, $L_i - L_j$ is not nef.

K-stability of \mathbb{P}^1 -bundles

Theorem (Ziquan Zhuang 2020)

Let X_1 and X_2 be two Fano varieties such that $X = X_1 \times X_2$. Then X is K -semistable (resp. K -polystable, K -stable, uniformly K -stable) if and only if X_1 and X_2 are both K -semistable (resp. K -polystable, K -stable, uniformly K -stable).

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Question

Study the K -stability of projective space bundles over a Fano variety.

Theorem (Kewei Zhang–Chuyu Zhou 2022)

Let Y be a Fano variety of dimension n and Fano index ≥ 2 . Take $L = -\frac{1}{r}K_Y$ for a rational number $r > 1$. Then $\mathbb{P}_Y(\mathcal{O}_Y \oplus L^\vee)$ is K -unstable.

\mathbb{P}^1 -bundles over G/P

Assume $k = 2$ and $L_1 = \mathcal{O}_Y$. Let $L = L_2$.

Corollary (Barban–H–Kwon)

Let $X = \mathbb{P}_Y(\mathcal{O}_Y \oplus L)$. Then

- $\text{Aut}^0(X)$ is reductive if and only if either L is trivial, or neither L nor $-L$ is nef on Y .
- In particular, if Y has Picard number 1, $\text{Aut}^0(X)$ is reductive if and only if $X \simeq Y \times \mathbb{P}^1$.

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Example

Let $Y = (\mathbb{P}^1)^n$, and consider the \mathbb{P}^1 -bundle $X = \mathbb{P}(\mathcal{O}_Y \oplus L)$, where $L = \mathcal{O}_Y(a_1, \dots, a_n)$, with $a_i \in \mathbb{Z}$ for every $i = 1, \dots, n$.

- X is Fano if and only if $a_i \in \{-1, 0, 1\}$ for every $i = 1, \dots, n$.
- $\text{Aut}^0(X)$ is reductive if and only if either $a_1 = \dots = a_n = 0$, or there exists $i \neq j$ such that $a_i a_j < 0$, for $i, j = 1, \dots, n$.

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Let Y be a rational homogeneous space, and let L be a nontrivial nef line bundle such that $-(K_Y + L)$ is ample. Then $\mathbb{P}_Y(\mathcal{O}_Y \oplus L^\vee)$ is a smooth K-unstable Fano variety.

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Thank you.